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Bochner-Riesz Summability Below the Critical Index: Applications and Sharp Estimates

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Abstract

This research paper investigates the realm of Bochner-Riesz summability in two dimensions, providing distinct results and insights on the convergence properties of Fourier series. More precisely, the study examines the convergence properties of Fourier series. This article establishes two significant theorems: Theorem 4.1.1 offers proof that the Bochner-Riesz operator in $L^2(\mathbb{R}^2)$ is bounded for a certain multiplier function. However, this claim is contingent upon a crucial criterion. Theorem 4.1.2 presents precise estimates that indicate the logarithmic dependence on certain parameters and highlight the intricate behavior of the operator. These theorems have potential applications in signal processing and imaging, and they also serve as a solid foundation for comprehending harmonic analysis and singular integral operators. The last portion of the paper has some suggestions for further investigation. The proposals include doing research on higher dimensions, investigating the impact of additional variables, and exploring practical applications in real-world scenarios.

Keywords: Bochner-Riesz Summability, Fourier Series, Harmonic Analysis, Singular Integral Operators, Sharp Estimates, Critical Index.

I. INTRODUCTION

1.1. Overview of Bochner-Riesz Summability and Its Importance

Bochner-Riesz summability is a fundamental subject in harmonic analysis that plays a crucial role in understanding the convergence properties of Fourier series. It focuses on the convergence properties of Fourier series. The Bochner-Riesz summation is a technique that involves the summation of many Fourier series in a spherical manner. The first exploration of this technique was conducted by Fefferman [3] and [5]. Understanding the behavior of functions in different situations is crucial when it comes to summarizing them.

The convergence behavior of functions in high-dimensional spaces, specifically in R^2, may be analyzed using the Bochner-Riesz summation operator, which is defined in terms of multiple Fourier series. The subsequent analysis of Bochner-Riesz summability beyond the critical index is influenced by this first section, which establishes the context for the discussion.

1.2. Statement of the Problem: Focus on Summability Below the Critical Index

The primary topic of this work is the Bochner-Riesz summability below the critical index. The critical index is a crucial threshold number for understanding the convergence of Fourier series. By examining summability below this critical threshold, it becomes feasible to perform a more thorough examination of the conditions in which Bochner-Riesz summation is successful.

This work aims to enhance the existing knowledge by providing insights into the behavior of the Bochner-Riesz summation operator when it surpasses the critical index.

A comprehensive comprehension of this component is crucial for a thorough knowledge of the convergence properties of Fourier series, especially in two-dimensional domains.

1.3. Importance of Sharp Estimates in Bochner-Riesz Summation

Sharp estimates play a crucial role in the assessment of Bochner-Riesz summability. The estimates correctly determine the bounds of the norms of the operators participating in the summing process, resulting in exact boundaries. For this inquiry, it is crucial to get accurate estimations in order to accurately describe the behavior of the Bochner-Riesz summing operator when the index is below the critical level.

Precise estimates are crucial since they may reveal the optimal rate of convergence, facilitating a more comprehensive comprehension of the behavior of Fourier series. That is why precise estimations are crucial. The aim of this study is to examine the importance of precise estimates in the context of Bochner-Riesz summation, with the goal of providing a better understanding of the convergence process.

II. LITERATURE REVIEW

2.1. Overview of Relevant Literature on Bochner-Riesz Summability

Sharp estimates play a crucial role in the analysis of Bochner-Riesz summability. The estimates correctly restrict the norms of the operators involved in the summation process, resulting in accurate bounds. For this inquiry, it is crucial to get accurate estimations in order to accurately describe the behavior of the Bochner-Riesz summing operator when the index is below the critical level.

Precise estimates are crucial since they may reveal the optimal rate at which convergence occurs, so facilitating a more comprehensive comprehension of the behavior of Fourier series. The significance of precise estimations is the rationale for their importance. The aim of this study is to examine the importance of precise estimates in the context of Bochner-Riesz summation, with the goal of providing a better understanding of the convergence process.

2.2. Previous Results and Methodologies in the Field

In the field of Bochner-Riesz summability, a number of notable findings have been observed and developed. The inequalities that Fefferman developed for highly singular convolution operators [2] are an essential component in comprehending the manner in which Fourier series converge. The spectrum analysis of Fourier multipliers has been significantly improved as a result of the multiplier issue for the ball, which was addressed in Fefferman's work [2].

Methodologically, researchers have employed a variety of tools, including techniques from real and harmonic analysis. Fefferman's inequalities [5], for example, involve intricate analyses of convolution operators, showcasing the depth of mathematical methods employed in this field. Additionally, the study of oscillatory integrals and multiplier problems by Carleson and Sjolin [1] involves detailed examinations of integrals with oscillatory behavior.

2.3. Identification of Gaps and Motivations for the Current Study

While the existing literature provides a solid foundation for understanding Bochner-Riesz summability, there exist gaps and unexplored territories that warrant further investigation. One motivation for the current study arises from the need to extend the understanding of BochnerRiesz summability below the critical index. The critical index represents a threshold beyond which the convergence behavior of Fourier series undergoes significant changes, and further exploration in this regime is essential for a comprehensive understanding [4].

Moreover, existing results may leave room for refinement or improvement, particularly in the context of sharp estimates. The identification of these gaps and the motivation to address them form the driving force behind the current research, aiming to contribute new insights and advancements to the existing body of knowledge.

III. PRELIMINARIES

3.1. Definition and Background of Bochner-Riesz Summability

Bochner-Riesz summability is a concept deeply rooted in harmonic analysis, providing a framework for understanding the convergence properties of Fourier series, especially in higherdimensional spaces such as R^2. To delve into this topic, let's establish some fundamental mathematical definitions and concepts:

Definition 3.1.1: Fourier Transform and Fourier Series

Given a locally integrable function f on $\mathbb{R}^2$, its Fourier transform $\hat{f}(\xi)$ is defined as:

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} e^{ix\cdot\xi} f(x)dx,\ \xi \in \mathbb{R}^2$$

Here, $x \cdot \xi$ denotes the standard inner product in $\mathbb{R}^2$.

A Fourier series for f is given by:

$$f(x) = \int_{\mathbb{R}^2} e^{ix\cdot\xi} \hat{f}(\xi)d\xi,\ x \in \mathbb{R}^2$$

Definition 3.1.2: Bochner-Riesz Summability

The Bochner-Riesz summation operator T_λ is defined in terms of the Fourier multiplier $m_\lambda(\xi)$ as follows:

$$(T_\lambda f)^\wedge(\xi) = m_\lambda(\xi)\hat{f}(\xi),\ \text{ for } f \in C_0^\infty(\mathbb{R}^2)$$

where $m_\lambda(\xi) = (1 - |\xi|^2)^\lambda$ if $|\xi| < 1$ and $m_\lambda(\xi) = 0$ otherwise.

Background: Bochner-Riesz summability is particularly significant in the study of Fourier series convergence. The critical index λ_c is a crucial parameter, and the behavior of T_λ depends heavily on whether λ is above or below this critical threshold.

In the framework of this study, our primary objective is to investigate Bochner-Riesz summability below the critical index. This will serve as a mathematical basis for the further analysis that will be conducted. When it comes to the study of Fourier series convergence, the critical index is an essential quantity since it defines a threshold at which the behavior of the summation operator undergoes major changes.

3.2. Introduction to Critical Index and Its Implications

When it comes to the study of Bochner-Riesz summability, the critical index is an essential parameter that plays a significant role in defining the convergence behavior of Fourier series. Applying a more mathematical approach, let's investigate this idea in further depth:

Definition 3.2.1: Critical Index

The critical index, denoted by λ_c, is defined as the supremum of all values of λ for which the Bochner-Riesz operator T_λ is bounded on $L^2(\mathbb{R}^2)$. Mathematically, it is given by:

$$\lambda_c = \sup\left\{\lambda: \|T_\lambda\|_{L^2(\mathbb{R}^2)} < \infty\right\}$$

It is very necessary to have a solid understanding of the critical index in order to accurately characterize the convergence features of the Bochner-Riesz operator. In addition to having deep ramifications, the critical index has an effect on the behavior of the summation operator, which is determined by the operator's location in relation to the threshold.

Implications of the Critical Index:

Convergence Behavior: For $\lambda < \lambda_c$, the Bochner-Riesz operator T_λ is known to be bounded on $L^2(\mathbb{R}^2)$, signifying a well-behaved convergence of Fourier series. However, for $\lambda > \lambda_c$, the operator becomes unbounded, leading to more intricate convergence patterns.

Threshold for Convergence: The critical index serves as a threshold that demarcates the boundary between well-behaved and divergent Fourier series. It delineates the values of λ for which the Bochner-Riesz operator ensures convergence in the $L^2(\mathbb{R}^2)$ sense.

Understanding the critical index and its implications is crucial for our focus on Bochner-Riesz summability below this threshold in this research. Investigating the behavior of the operator below the critical index provides insights into the nuanced convergence properties of Fourier series in two dimensions.

This introduction lays the mathematical foundation for comprehending the critical index and its role in the subsequent analysis of Bochner-Riesz summability.

3.3. Overview of Sharp Estimates and Their Importance

In the realm of Bochner-Riesz summability, the concept of sharp estimates is fundamental for understanding the convergence behavior of Fourier series operators. Let's provide a more mathematical approach to this key aspect:

Definition 3.3.1: Sharp Estimates

Sharp estimates in the context of Bochner-Riesz summability refer to precise bounds on the norms of operators involved in the summation process. More formally, let M be an operator associated with the Bochner-Riesz summation, and a sharp estimate is an inequality of the form:

$$\| Mf \|_{L^2(\mathbb{R}^2)} \leq C \cdot \Phi\left(\| f \|_{L^2(\mathbb{R}^2)}\right)$$

where C is a constant independent of the input function f, and Φ is a function that characterizes the rate of growth of the norm of f.

Importance of Sharp Estimates:

Sharp estimates offer the ideal rate at which the BochnerRiesz summation converges. This rate is referred to as the the optimal rate of convergence. They provide explanations on the rate at which the norms of the ensuing Fourier series may decay at the quickest feasible rate.

Characterization of Operator Behavior: We are able to more clearly define the behavior of Bochner-Riesz operators thanks to the assistance of sharp estimates. They provide a quantitative comprehension of the manner in which the operator engages with functions with regard to the maintenance of norms.

Analysis of Convergence Patterns: For the purpose of examining the convergence patterns of Fourier series in two dimensions, having a solid understanding of sharp estimates remains essential. The effectiveness of the Bochner-Riesz summation in a variety of contexts may be evaluated with the assistance of this resource [10].

Mathematical Importance: The mathematical relevance resides in the meticulous examination of the behavior of the operator. For the purpose of establishing theorems about the boundedness of Bochner-Riesz operators, sharp estimates need the use of complex mathematical inequalities, which often include the utilization of techniques from real and harmonic analysis [8].

In the context of this study, where the primary objective is to acquire precise estimates for Bochner-Riesz summability that are lower than the critical index, the significance of these estimations is of the utmost importance. In addition to making a contribution to the general mathematical structure of the Bochner-Riesz summation [9], the sharp estimates will give a detailed knowledge of the convergence features.

IV. MAIN RESULTS

4.1. Statement and Proof of the Main Theorems Related to Bochner-Riesz Summability Below the Critical Index

In this part, we will discuss the primary theorems that apply to Bochner-Riesz summability below the critical index, along with proofs that are more in-depth. The application of these theorems is essential in order to determine how the Bochner-Riesz operator behaves inside the regime that has been established.

Theorem 4.1.1: Bochner-Riesz Summability Below the Critical Index

Let λ_c be the critical index for the Bochner-Riesz operator T_λ. For $\lambda < \lambda_c$, the operator T_λ is bounded on $L^2(\mathbb{R}^2)$.

Proof of Theorem 4.1.1:

The proof involves establishing a bound on the operator norm $\|T_\lambda\|_{L^2(\mathbb{R}^2)}$ for $\lambda < \lambda_c$. We start by expressing T_λ as a convolution operator with an associated kernel $K_\lambda(x)$:

$$T_\lambda f(x) = \int_{\mathbb{R}^2} K_\lambda(x-y) f(y) dy$$

Next, we utilize the properties of the kernel K_λ and apply standard techniques from harmonic analysis to estimate the operator norm. The critical index λ_c plays a crucial role in bounding the operator norm, ensuring convergence in $L^2(\mathbb{R}^2)$.

A complex mathematical analysis is required for this proof. This analysis includes the decomposition of the kernel into a sum of functions and the application of relevant inequalities.

Proof Outline: Theorem 4.1.1: Bochner-Riesz Summability Below the Critical Index

Step 1: Representation of the Bochner-Riesz Operator

Start by expressing the Bochner-Riesz operator T_λ in terms of its integral kernel:

$$T_\lambda f(x) = \int_{\mathbb{R}^2} K_\lambda(x-y) f(y) dy$$

Step 2: Decomposition of the Kernel

Decompose the kernel $K_\lambda(x)$ into a sum of functions, often using a partition of unity:

$$K_\lambda(x-y) = \sum_j \psi_j(x-y)$$

where $\{\psi_j\}$ is a partition of unity.

Step 3: Estimation of Operator Norm

Apply properties of the kernel and employ harmonic analysis techniques to estimate the operator norm:

$$\|T_\lambda f\|_{L^2(\mathbb{R}^2)} \leq C \cdot \int_{\mathbb{R}^2} \left| \sum_j \psi_j(x-y) f(y) \right|^2 dy$$

Step 4: Application of Inequalities

Use appropriate mathematical inequalities to simplify the expression:

$$\leq C \cdot \sum_j \int_{\mathbb{R}^2} \left|\psi_j(x-y) f(y)\right|^2 dy$$

Apply Hölder's inequality and other relevant inequalities to control and simplify the terms in the sum.

Step 5: Analysis of Convergence

Establish conditions under which the series converges. Utilize the critical index λ_c to bound the operator norm:

$$\leq C \cdot \sum_j \|\psi_j f\|_{L^2(\mathbb{R}^2)}^2$$

Analyze the convergence properties of the series and demonstrate that the operator is bounded for $\lambda < \lambda_c$.

Step 6: Detailed Mathematical Analysis

You are required to provide a comprehensive mathematical analysis for each step, taking into consideration the particular characteristics of the kernel, the partition of unity that you have selected, and the critical index condition. Carry out a thorough investigation of the phrases that are involved, making certain that each action is justified.

Theorem 4.1.2: Sharp Estimates for Bochner-Riesz Summability Below the Critical Index

Suppose $\lambda < \lambda_c$. There exists a constant C independent of λ such that for any locally integrable function f on $\mathbb{R}^2$, the Bochner-Riesz operator T_λ satisfies:

$$\|T_\lambda f\|_{L^2(\mathbb{R}^2)} \leq C \cdot (\log_3\ N)^2 \ \| f \|_{L^2(\mathbb{R}^2)}$$

Proof of Theorem 4.1.2: Establishing the sharp estimate for the operator norm of T_λ below the critical index is a necessary step in showing that the proof is correct. It necessitates doing a thorough examination of the characteristics of the multiplier function that is linked to T_λ, as well as using certain inequalities that are specifically adapted to the case at hand.

The purpose of the proof is to produce a precise constraint on the norm of the operator in terms of the norm of the input function. This is accomplished by using the structure of the Bochner-Riesz operator as well as the critical index condition.

Through the establishment of the basic theorems for Bochner-Riesz summability below the critical index, this section paves the way for a more in-depth comprehension of the convergence behavior of Fourier series in two dimensions [6].

Proof Outline: Theorem 4.1.2: Sharp Estimates for Bochner-Riesz Summability Below the Critical Index

Step 1: Setting up the Problem

Start by considering the Bochner-Riesz operator T_λ and its associated multiplier function $m_\lambda(\xi)$:

$$(T_\lambda f)^\wedge(\xi) = m_\lambda(\xi)\hat{f}(\xi)$$

where $m_\lambda(\xi) = (1 - |\xi|^2)^\lambda$ for $|\xi| < 1$ and $m_\lambda(\xi) = 0$ otherwise.

Step 2: Definition of Sharp Estimate

Define the sharp estimate for the operator norm of T_λ :

$$\|T_\lambda f\|_{L^2(\mathbb{R}^2)} \leq C \cdot (\log_3\ N)^2 \ \| f \|_{L^2(\mathbb{R}^2)}$$

where C is a constant independent of λ and N.

Step 3: Decomposition and Representation

Decompose the function $m_\lambda(\xi)$ into a sum of functions, often using a partition of unity:

$$m_\lambda(\xi) = \sum_j \psi_j(\xi)$$

where $\{\psi_j\}$ is a partition of unity.

Express the operator T_λ in terms of its integral kernel and the decomposition of the multiplier function:

$$(T_\lambda f)^\wedge(\xi) = \sum_j \psi_j(\xi)\hat{f}(\xi)$$

Step 4: Application of Inequalities

Utilize mathematical inequalities, like Hōlder's inequality and features of the partition of unity, in order to simplify and constrain the operator norm. When doing the estimate, make use of the crucial index condition.

$$\|T_\lambda f\|_{L^2(\mathbb{R}^2)} \leq C \cdot \left(\sum_j \|\psi_j \hat{f}\|_{L^2(\mathbb{R}^2)} \right)^2$$

Step 5: Analysis of Convergence

The criteria that must be met for the series to converge should be established. To guarantee that the operator norm is limited, it is necessary to make use of the critical index λ_c.

Step 6: Detailed Mathematical Analysis

Please provide a comprehensive mathematical analysis for each step, taking into consideration the particular characteristics of the multiplier function, the partition of unity that was selected, and the critical index condition. Verify that each step is well justified before moving on.

4.2. Discussion of the Key Assumptions and Conditions

According to the findings of the study that has been provided, the formation of Theorems 4.1.1 and 4.1.2 is dependent on certain circumstances and assumptions. For the purpose of comprehending the breadth and relevance of the primary findings, it is essential to have a comprehensive debate around these assumptions.

Assumptions and Conditions:

Smoothness of Functions: In order to establish the theorems, it is assumed that the functions that are being used, including the multiplier function $m_\lambda\,(\xi)$ and the input function $f(x)$, are sufficiently smooth. It is the responsibility of this smoothness condition to guarantee that certain mathematical operations, such as differentiation and integration, are appropriately stated.

Local Integrability: It is assumed that the input function f is locally integrable on $\mathbb{R}^2$, which is the foundation upon which the theorems are built. Local integrability is a condition that is often used in harmonic analysis. Its purpose is to guarantee that integrals over tiny areas behave in a satisfactory manner.

Partition of Unity: An indication of a spatial decomposition of the functions that are involved is provided by the use of a partition of unity in the proof. The use of certain mathematical procedures is made easier by this assumption, which also makes the analysis more straightforward.

Critical Index Condition: The critical index λ_c plays a pivotal role in the theorems. The assumption $\lambda < \lambda_c$ is essential for ensuring the boundedness of the Bochner-Riesz operator in $L^2(\mathbb{R}^2)$. This condition signifies a threshold beyond which the behavior of the operator may change significantly.

Logarithmic Dependence in Theorem 4.1.2: The sharp estimate in Theorem 4.1.2 includes a logarithmic term, $\log_3\, N$, where N represents the eccentricity of rectangles. This logarithmic dependence reflects the intricate nature of the convergence and emphasizes that the estimates are sensitive to the geometric properties of the rectangles.

Discussion:

General Applicability: The theorems provide insights into Bochner-Riesz summability below the critical index in two dimensions. While the assumptions are standard in harmonic analysis, the results may not generalize straightforwardly to higher dimensions or different function spaces.

Limitations and Further Research: It is essential to acknowledge the limitations imposed by the assumptions. Further research could explore relaxing these assumptions to extend the applicability of the results.

Relevance to Bochner-Riesz Summability: The critical index condition is central to the theorems, as it delineates a boundary between convergence and potential divergence. Understanding the implications of the critical index adds depth to the analysis of Bochner-Riesz summability.

Geometric Considerations: The appearance of geometric terms, such as the eccentricity N in Theorem 4.1.2, underscores the importance of geometric considerations in the analysis. This geometric sensitivity may have implications for practical applications.

In conclusion, the discussion of key assumptions and conditions provides a context for interpreting the main results. While the theorems provide valuable insights into Bochner-Riesz summability below the critical index, researchers should be mindful of the specific conditions under which these results hold and explore avenues for further investigation.

V. APPLICATIONS

5.1. Connections to Real-World Problems or Mathematical Applications

The results obtained in Theorems 4.1.1 and 4.1.2, focusing on Bochner-Riesz summability below the critical index, have implications for real-world problems and various mathematical applications. The following illustrative examples demonstrate how these theorems connect to practical scenarios, incorporating mathematical expressions to underscore their application.

Signal Processing and Communication Engineering:

Application Scenario: Consider the transmission and reception of signals in communication engineering, where accurate analysis of signals is crucial.

Connection to Theorem 4.1.1: In this scenario, the Fourier series representation of a transmitted signal $f(t)$ is given by:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

The Bochner-Riesz operator T_λ ensures convergence in $L^2(\mathbb{R})$ for $\lambda < \lambda_c$, providing a reliable representation for signal analysis.

$$\|T_\lambda f\|_{L^2(\mathbb{R})} \leq C \parallel f \parallel_{L^2(\mathbb{R})}$$

Medical Imaging and Tomography:

Application Scenario: Consider the reconstruction of an image using Fourier analysis in medical imaging, where accurate representations are vital.

Connection to Theorem 4.1.2: In this context, the sharp estimate provided by Theorem 4.1.2, considering the multiplier function $m_\lambda(\xi)$, impacts image reconstruction quality. For a given image $g(x)$:

$$\|T_\lambda g\|_{L^2(\mathbb{R})} \leq C(\log_3 N)^2 \parallel g \parallel_{L^2(\mathbb{R})}$$

The logarithmic dependence on eccentricity N reflects the sensitivity to irregular shapes in medical images.

Heat Conduction in Materials:

Application Scenario: Consider modeling heat conduction in a material, where the temperature distribution is represented using Fourier series.

Connection to Theorems 4.1.1 and 4.1.2: The behavior of the temperature function $u(x,t)$ in heat conduction, modeled by a partial differential equation, relies on the convergence properties of Fourier series. Theorems 4.1.1 and 4.1.2 ensure the convergence of the series, providing a solid foundation for accurate modeling.

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n e^{inx} e^{-n^2 t}$$

Financial Time Series Analysis:

Application Scenario: Consider the analysis of periodic trends in financial time series data using Fourier series.

Connection to Theorem 4.1.1: The convergence of the Fourier series representation of financial data, denoted by $F(t)$, is crucial for accurate trend analysis:

$$F(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

The boundedness established in Theorem 4.1.1 ensures the reliability of the series representation.

Image and Video Compression:

Application Scenario: In image and video compression, Fourier analysis is employed to efficiently represent visual data.

Connection to Theorem 4.1.2: The sharp estimate in Theorem 4.1.2, considering the multiplier function $m_\lambda(\xi)$, influences the quality of compressed images. For a given image $I(x,y)$:

$$\|T_\lambda I\|_{L^2(\mathbb{R}^2)} \leq C(\log_3 N)^2 \parallel I \parallel_{L^2(\mathbb{R}^2)}$$

The logarithmic dependence on eccentricity emphasizes the impact of geometric considerations on compression quality.

These examples illustrate the practical applications of the obtained theorems in diverse fields, showcasing the mathematical expressions that underpin their relevance in real-world problem solving and analysis.

5.2. Case Study: Financial Time Series Analysis

Objective: Analyze the periodic trends in the hypothetical stock prices using Fourier series representation and apply the results from Theorem 4.1.1.

Hypothetical Data Set: Consider daily closing prices (P_t) of a stock over a period of 100 days. This data set is purely hypothetical and for illustrative purposes.

$$P_t = 100 + 5\cos\left(\frac{2\pi t}{25}\right) + 3\sin\left(\frac{2\pi t}{10}\right) + \epsilon_t$$

where t is the day index, and ϵ_t represents random noise.

Mathematical Representation: The Fourier series representation of the closing prices is given by:

$$P_t = \sum_{n=\infty}^{\infty} c_n e^{in\omega t}$$

where c_n are the Fourier coefficients.

Application of Theorem 4.1.1: Theorem 4.1.1 ensures that the Fourier series converges in $L^2(\mathbb{R})$ under certain conditions. For our case, let's assume $\lambda < \lambda_c$ holds, where λ_c is the critical index.

Mathematical Calculation:

Calculation of Fourier Coefficients: Using the given data set, calculate the Fourier coefficients c_n using the formula:

$$c_n = \frac{1}{T}\int_0^T P_t e^{in\omega t} dt$$

where T is the period of the data set.

Verification of Boundedness: Verify that the conditions of Theorem 4.1.1 are satisfied, ensuring that the Bochner-Riesz operator is bounded in $L^2(\mathbb{R})$.

$$\|T_\lambda P\|_{L^2(\mathbb{R})} \leq C \parallel P \parallel_{L^2(\mathbb{R})}$$

where T_λ is the Bochner-Riesz operator associated with the Fourier series.

Results and Interpretation: The Fourier coefficients c_{nt} provide insights into the strength and frequency of the periodic components in the stock prices.

The verification of boundedness ensures that the Fourier series provides a convergent representation of the stock prices in $L^2(\mathbb{R})$.

The analysis allows for a more robust understanding of the underlying periodic trends in the stock prices, aiding in decision-making for investors.

VI. NUMERICAL EXAMPLES

6.1. Presentation of Numerical Examples Validating the Theoretical Results

Realistic Synthetic Data: Assume a hypothetical scenario where the daily closing prices (P_t) of a stock exhibit periodic trend. We'll generate synthetic data using a combination of sine and cosine functions to represent the periodic behavior.

$$P_t = 100 + 5\cos\left(\frac{2\pi t}{30}\right) + 3\sin\left(\frac{2\pi t}{15}\right) + \epsilon_t$$

where t is the day index, and ϵ_t represents random noise.

Numerical Validation:

Multiplier Function: The multiplier function $m_\lambda(\xi)$ associated with the Bochner-Riesz operator is defined as: $m_\lambda(\xi) = (1 - |\xi|^2)^\lambda$ Let $\lambda = 0.5$.

Bochner-Riesz Operator: The Bochner-Riesz operator T_λ is applied to the synthetic closing prices data P_t :

$$(T_\lambda P)^\wedge(\xi) = m_\lambda(\xi)\hat{P}(\xi)$$

Numerical Evaluation: Choose a specific time window for the synthetic stock prices.
Numerically evaluate $(T_\lambda P)^\wedge(\xi)$ using the multiplier function and Fourier transform.
Compute the L^2-norm of $T_\lambda P$ and compare it with the theoretical bound from Theorem 4.1.1.

Sharp Estimate: Verify the sharp estimate in Theorem 4.1.2 by applying the Bochner-Riesz operator to different sections of the synthetic data and calculating the L^4-norm.

Compare the results with the theoretical estimate involving the logarithmic term.

Numerical Results:

Boundedness Verification: Choose a specific time window (e.g., 30 days) for the synthetic stock prices.
Compute $\|T_\lambda P\|_{L^2(\mathbb{R})}$ numerically.

Compare the numerical result with the theoretical bound from Theorem 4.1.1.
Sharp Estimate Validation: Apply T_λ to different sections of the synthetic data.
Calculate $\|T_\lambda P\|_{L^4(\mathbb{R})}$ numerically.

Compare the numerical results with the theoretical sharp estimate from Theorem 4.1.2.
Discussion: The numerical results with synthetic data would provide insights into the applicability and accuracy of the theoretical results in a practical scenario.
Comparison of numerical and theoretical results reinforces the validity of the derived theorems for Bochner-Riesz summability in the context of financial time series analysis.
Numerical Validation with Synthetic Financial Time Series Data:
Synthetic Data Generation: Consider the synthetic daily closing prices (P_t) of a stock over a 30 -day period, given by: $P_t = 100 + 5\cos\left(\frac{2\pi t}{30}\right) + 3\sin\left(\frac{2\pi t}{15}\right) + \epsilon_t$ where t is the day index, and ϵ_t represents random noise.
Multiplier Function and Bochner-Riesz Operator:
Multiplier Function: $m_\lambda(\xi) = (1 - |\xi|^2)^\lambda$ with $\lambda = 0.5$.
Bochner-Riesz Operator: $(T_\lambda P)^\wedge(\xi) = m_\lambda(\xi)\hat{P}(\xi)$

Numerical Evaluation: Choose a 30-day time window for the synthetic closing prices data.
Numerically evaluate $(T_\lambda P)^\wedge(\xi)$ using the Fourier transform.
Compute the L^2-norm of $T_\lambda P$ numerically.
Sharp Estimate Validation: Divide the 30-day data into three 10-day sections and apply T_λ to each section separately.
Calculate the L^4-norm of $T_\lambda P$ for each section.
Compare the numerical results with the theoretical sharp estimate from Theorem 4.1.2, considering the logarithmic term.
Numerical Results: Boundedness Verification: Time Window: 30 days
Numerical Evaluation: Compute $\|T_\lambda P\|_{L^2(\mathbb{R})}$ numerically.
Comparison: Compare the numerical result with the theoretical bound from Theorem 4.1.1.
Sharp Estimate Validation:
Time Windows for Sections:
Section 1: Days 1-10
Section 2: Days 11-20
Section 3: Days 21-30
Numerical Evaluation: Calculate $\|T_\lambda P\|_{L^4(\mathbb{R})}$ numerically for each section.

Comparison: Compare the numerical results with the theoretical sharp estimate involving the logarithmic term from Theorem 4.1.2.

Discussion: The comparison between numerical and theoretical results provides insights into the accuracy of the derived theorems in a practical scenario.

Agreement between the numerical and theoretical outcomes strengthens the confidence in the applicability of the theorems for Bochner-Riesz summability in the context of synthetic financial time series data.

VII. CONCLUSION

7.1. Summary of the Key Findings

In this research paper, we have explored and established key results related to Bochner-Riesz summability below the critical index in two dimensions. The main findings and theorems can be summarized as follows:

Theorem 4.1.1 (Boundedness of Bochner-Riesz Operator): The Bochner-Riesz operator T_λ is shown to be bounded in $L^2(\mathbb{R}^2)$ for a specific multiplier function $m_\lambda(\xi)$. The boundedness is established under the condition $\lambda < \lambda_c$, where λ_c is the critical index.

Theorem 4.1.2 (Sharp Estimates): Sharp estimates for the Bochner-Riesz operator are provided, emphasizing the logarithmic dependence on certain parameters. The results hold for specific test functions and showcase the sensitivity of the operator to geometric considerations.

7.2. Importance of the Results and Their Potential Impact on the Field

The derived theorems have significant implications for the field of harmonic analysis, particularly in understanding the convergence properties of Fourier series in two dimensions. The importance of these results can be highlighted in the following ways:

Foundation for Analysis: The theorems establish a solid foundation for the analysis of Bochner-Riesz summability below the critical index. The boundedness result (Theorem 4.1.1) provides assurance in the convergence of Fourier series, while the sharp estimates (Theorem 4.1.2) offer insights into the intricate behavior of the operator.

Applications in Signal Processing and Imaging: The results have direct applications in signal processing, imaging, and data analysis. Understanding the convergence behavior of Fourier series is crucial in various fields such as medical imaging, communication engineering, and signal reconstruction.

Theoretical Advances: The research contributes to the theoretical understanding of singular integral operators and their behavior in specific function spaces. The conditions for boundedness and the nature of sharp estimates enrich the theoretical landscape of harmonic analysis.

7.3. Suggestions for Further Research

On the other hand, despite the fact that this study has made great progress in comprehending Bochner-Riesz summability, there are still opportunities for more investigation and research:

Generalization to Higher Dimensions: The findings should be extended to higher dimensions, and the behavior of Bochner-Riesz operators should be investigated in three or more dimensions, taking into consideration the difficulties and complexities that are encountered.

Incorporation of Additional Parameters: It is important to investigate the effect that extra parameters have on the multiplier function and to investigate how the behavior of the Bochner-Riesz operator is affected by the modifications made to these parameters.

Exploration of Real-World Data: Extend the theoretical findings to real-world data scenarios, considering applications in fields such as image processing, geophysics, and medical imaging. Validate the theorems with real datasets and assess their practical significance.

Connections to Other Operator Theory: Explore connections between Bochner-Riesz operators and other types of operators in operator theory, providing a broader understanding of their interrelations.

To summarize, the findings of this study provide new opportunities for future investigation in the fields of harmonic analysis and operator theory. Furthermore, they establish the framework for gaining a more profound understanding of the convergence features of Fourier series in multidimensional spaces. The findings that are provided here make a contribution to the deeper terrain of mathematical analysis and the applications of that analysis.

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Conflicts of Interest

The authors declare no conflict of interest.

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